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On the ill conditioning of locating transmission zeros in least squares ARMA filtering

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Abstract: The algorithms of Levinson–Schur and Nevanlinna–Pick are briefly reviewed. Both produce least squares predictive filters. By minimizing the least squares error with respect to the interpolation points of the Nevanlinna–Pick algorithm we find the transmission zeros of an ARMA filter. It is shown by some simple examples that this is an ill conditioned problem.

Keywords: Rational approximation, digital filters, transmission zeros.

1. Introduction

The Levinson algorithm is well known in the literature of linear predictive filtering. It constructs autoregressive (AR) filters from the spectral information of the unknown filter. Dual to the Levinson algorithm is the Schur algorithm which, like the Levinson algorithm constructs a sequence of reflection coefficients. The problem solved by these two algorithms is a weighted least squares rational approximation. However, the numerator is fixed to be a constant. It is possible to extend the Schur algorithm to generate autoregressive-moving average (ARMA) filters. This generalization is known as the Nevanlinna–Pick algorithm. It solves the same type of rational approximation problem, but now with a fixed numerator polynomial that need not be a constant. This numerator is defined by its zeros, which are called the transmission zeros (TZ) of the filter. These transmission zeros are important because they define the interpolation points for the Nevanlinna–Pick algorithm. Thus for any choice of genuine transmission zeros this algorithm will give an optimal least squares approximant. Thus the least squares error is a nonnegative function of these TZ. The optimal choice of the TZ would then be the one that minimizes this least squares error. In this paper we shall show by some simple examples that the problem of finding the exact TZ of the given filter in this way is a very ill conditioned one. If we turn this argument around it means that if the chosen TZ are far away from the exact ones, the Nevanlinna–Pick algorithm can still give an ARMA filter which is a very good least squares approximant. The practical consequence of this is that it is sufficient to make only a coarse guess of the TZ to construct the ARMA filter.

In Section 2 and 3 we briefly review the Levinson–Schur and the Nevanlinna–Pick extraction algorithms. Section 2 is not essential for our result, but it reviews the methods best known in the literature of linear predictive filtering. It is included because the Nevanlinna–Pick algorithm can then be introduced as a simple generalization of it. Section 4 introduces the optimization problem for the optimal location of the TZ and finally Section 5 will give some examples illustrating the ill conditionering of the problem.

2. The classical Szegö–Levinson–Schur extraction algorithms

Let $x(t)$, $t \in \mathbb{Z}$, be a real stationary zero mean scalar stochastic sequence with covariance function

$$R_k = E[x(t)x(t-k)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\omega(e^{i\theta})\, d\theta, \tag{2.1}$$

and ω the spectral density. Before we give a first theorem on the spectral factorization of ω we introduce the notations T for the unit circle: $T = \{z\colon |z| = 1\}$, D for the open unit disc: $D = \{z\colon |z| < 1\}$. L_p are the classical Hilbert spaces for the unit circle and H_p the corresponding Hardy spaces. Finally by $f_*(z)$ we mean $\overline{f(1/\bar{z})}$.

Theorem 2.1. *Let* $\omega(e^{i\theta}) = \omega(e^{-i\theta})$ *(even),* $\omega(e^{i\theta}) \geqslant 0$ *(nonnegative),* ω *and* $\log \omega \in L_1$. *Then there exists a spectral factor* σ *which is outer in* H_2 [14], *such that*

$$\omega(e^{i\theta}) = |\sigma(e^{i\theta})|^2, \tag{2.2}$$

which is given by

$$\sigma(e^{i\theta}) = c \cdot \lim_{r \uparrow 1} \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\psi} + re^{i\theta}}{e^{i\psi} - re^{i\theta}} \log \omega(e^{i\psi})\, d\psi\right\},$$

c is a constant of modulus 1. If we take $c = 1$, then the Taylor coefficients of $\sigma(z)$ are real and $\sigma(0) > 0$. Under roomyness conditions [8], *ω and σ may be defined in the whole complex plane and* (2.2) *then generalizes to*

$$\omega(z) = \sigma(z)\sigma_*(z).$$

The function

$$\Omega(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}}\omega(e^{i\theta})\, d\theta = R_0 + 2\sum_{k=1}^{\infty} R_k z^k,$$

is a Carathéodory function, i.e., it is analytic in D and its real part is nonnegative there.

Proof. See [18]. □

It is a classical result that if $\Omega(z)$ is a Carathéodory function, then $\Gamma_0(z) = (\Omega(z) - R_0)/(\Omega(z) + R_0)$ is a Schur function. I.e. Γ_0 is analytic in D and $|\Gamma_0(z)| \leqslant 1$ in this region [1].

The Schur algorithm constructs a set of reflection coefficients ρ_k, $k = 0, 1, \ldots,$ as follows [1,15]:

$$\Gamma_k'(z) = \Gamma_k(z)/z, \qquad \rho_k = \Gamma_k'(0) = \lim_{z \to 0} \Gamma_k(z)/z, \qquad \Gamma_{k+1}(z) = \frac{\Gamma_k'(z) - \rho_k}{1 - \bar{\rho}_k \Gamma_k'(z)},$$

for $k = 0, 1, 2, \ldots$

On this algorithm we have the following:

Theorem 2.2. *$\Gamma_0(z)$ is a Schur function iff*

(1) $|\rho_k| < 1$, $k = 1, 2, \ldots,$ or

(2) $|\rho_k| < 1$, $k = 1, 2, \ldots, n$, and $|\rho_{n+1}| = 1$. In this case Γ_0 is a Blaschke product of degree n, $\Gamma_{n+1}(z) = \rho_{n+1}$ and $\Gamma_k(z) = \rho_k = 0$ for all $k > n + 1$.

In both cases all Γ_k are Schur functions that vanish at the origin.

Proof. See [1,15]. □

This algorithm of Schur can be reformulated in an homogeneous way. We define

$$\Delta_1^0(z) = \Omega(z) - R_0, \qquad \Delta_2^0(z) = \Omega(z) + R_0, \qquad \Delta^0(z) = \tfrac{1}{2}\left[\Delta_1^0(z) \quad \Delta_2^0(z)\right].$$

We call such an expression $\Delta(z) = [\Delta_1(z)\ \Delta_2(z)]$ an acceptable function if Δ_1 and Δ_2 are analytic in D and if Δ_1/Δ_2 is a Schur function. Clearly $\Delta^0(z)$ is acceptable. The Schur algorithm is now equivalent with

$$\Delta^{k\prime}(z) = \Delta^k(z)\begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \qquad \rho_k = \Delta_1^{k\prime}(0)/\Delta_2^{k\prime}(0),$$

$$\Delta^{k+1}(z) = \Delta^{k\prime}(z)\left(1 - |\rho_k|^2\right)^{-1/2}\begin{bmatrix} 1 & -\bar{\rho}_k \\ -\rho_k & 1 \end{bmatrix}, \qquad \text{for } k = 1, 1, 2, \ldots,$$

which is the same as

$$\Delta^{k+1}(z) = \Delta^k(z)\theta_k^{-1}(z),$$

with

$$\theta_k(z) = \left(1 - |\rho_k|^2\right)^{-1/2}\begin{bmatrix} 1 & \bar{\rho}_k \\ \rho_k & 1 \end{bmatrix}\begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}.$$

It follows directly from previous theorem that all Δ^k are acceptable and $\Delta_1^k(0) = 0$.

Define recursively

$$\Theta_0 = R_0^{-1/2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Theta_{n+1} = \theta_n \Theta_n \quad \text{for } n \geqslant 0.$$

Then Θ_n has the following structure:

Theorem 2.3.

$$\Theta_n = \tfrac{1}{2}\begin{bmatrix} \phi_n + \psi_n & \phi_n - \psi_n \\ \phi_n^* - \psi_n^* & \psi_n^* + \psi_n^* \end{bmatrix},$$

where $\phi_k(z)$ are the Szegö orthonormal polynomials [16]

$$(\phi_k, \phi_l)_\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} \phi_k(e^{i\theta})\overline{\phi_l(e^{i\theta})}\omega(e^{i\theta})d\theta = \delta_{kl}$$

and $\psi_k(z)$ are the associated polynomials of the second kind [9].

The upper star denotes the parahermitian conjugate. For a polynomial of degree n it is defined as $\phi^*(z) = z^n\phi_*(z)$.

Proof. See [7,8]. □

It directly follows that we have the following recursion for these polynomials:

$$\begin{bmatrix}\phi_{n+1}(z)\\ \phi^*_{n+1}(z)\end{bmatrix} = \theta_n\begin{bmatrix}\phi_n(z)\\ \phi^*_n(z)\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}\psi_{n+1}(z)\\ -\psi^*_{n+1}(z)\end{bmatrix} = \theta_n\begin{bmatrix}\psi_n(z)\\ -\psi^*_n(z)\end{bmatrix},$$

with initial conditions $\phi_0 = \psi_0 = R_0^{-1/2}$. Clearly

$$\phi^*_{n+1}(0) = \kappa_{n+1} = \left(R_0\prod_{k=1}^{n}\left(1-|\rho_k|^2\right)\right)^{-1/2} = \kappa_n\left(1-|\rho_n|^2\right)^{-1/2}$$

and

$$\rho_n = \overline{\phi_{n+1}(0)}/\kappa_{n+1} = -\left(\phi^*_n(z), z\phi_n(z)\right)_\omega = -\kappa_n\sum_{k=0}^{n}\bar{q}_k R_{k+1},$$

where $\phi_n(z) = \sum_{k=0}^n q_k z^k$ and $q_n = \kappa_n$.

This way of computing the reflection coefficients is known as the Levinson algorithm or Szegö's recursion and in the context of linear algebra as the Trench–Zohar–Rissanen factorization method for Toeplitz matrices [17,19,13].

From [5,9] we obtain e.g. the following Padé-like approximants.

Theorem 2.4. (1) *Define $\tilde{\Omega}(z) = \Omega(z)/R_0$ and $\tilde{\Omega}_n(z) = \psi^*_n(z)/\phi^*_n(z)$. Then the McLaurin series of $\tilde{\Omega}(z)$ and $\tilde{\Omega}_n(z)$ correspond up to the term in z^n. This means that $\tilde{\Omega}_n(z)$ is a Padé type approximant of $\tilde{\Omega}(z)$* [2].

(2) *Define $\tilde{\omega}(z) = \omega(z)/R_0$ and $\tilde{\omega}_n(z) = \frac{1}{2}[\tilde{\Omega}_n(z) + \tilde{\Omega}_{n*}(z)]$. Then $\tilde{\omega}_n(z) = R_0^{-1}z^n/(\phi_n(z)\,\phi^*_n(z))$ and the Fourier series of $\tilde{\omega}_n(z)$ has the form*

$$\tilde{\omega}_n(e^{i\theta}) = \sum_{-\infty}^{\infty} r_k e^{ik\theta} \quad \text{with } r_k = R_0^{-1}R_k,\ k = 0, \pm 1, \pm 2, \ldots, \pm n,$$

which means that $\tilde{\omega}_n(z)$ is a $(0, n)$ Laurent–Padé approximant of $\tilde{\omega}(z)$ [5].

Proof See [5]. □

Most important is that we also have a best approximant of σ^{-1} in the Hilbert space $L_2(\omega)$, i.e. the space of square integrable functions on the unit circle with weight ω. Indeed we have

Theorem 2.5. *The minimum* S_n *of*

$$\min_{p_n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{\sigma(e^{i\theta})} - p_n(e^{i\theta}) \right|^2 \omega(e^{i\theta})\, d\theta,$$

where p_n *ranges over all polynomials of degree* n *is obtained for*

$$p_n(z) = \sigma(0)\kappa_n\phi_n^*(z)$$

and this minimum is given by

$$S_n = 1 - \kappa_n^2\sigma(0)^2.$$

Proof. See [9]. □

3. The Nevanlinna–Pick algorithm

Theorem 2.5 gave the projection in $L_2(\omega)$ of $(\sigma(0)\,\sigma(z))^{-1}$ on span $\{1, z, \dots, z^n\}$, which only gives polynomial approximations.

A polynomial can be considered as a rational function with all its poles at infinity. Therefore the previous problem is a special case of the following: what is the projection of $(\sigma(0)\,\sigma(z))^{-1}$ in $L_2(\omega)$ on the space $\mathscr{L}_n = \operatorname{span}\{U_0(z), U_1(z), \dots, U_n(z)\}$, where the functions $\{U_k(z)\}_0^n$ form a basis of the space of rational functions with poles $1/\alpha_n$, $k = 1, 2, \dots, n$. We shall take for $U_k(z)$ the partial Blaschke products:

$$U_0 = 1 \quad\text{and}\quad U_k = U_{k-1}\zeta_k, \quad \text{with } \zeta_k = (z-\alpha_k)/(1-\bar{\alpha}_k z),\ k = 1, 2, \dots,$$

where all α_k are in D.

To solve this problem, the Schur algorithm is extended to the Nevanlinna–Pick algorithm [12,18] which goes as follows.

Take $\Gamma_0(z)$ as before and then

$$\gamma_k = \begin{cases} \Gamma_k(\alpha_{k+1}) & \text{for } \alpha_{k+1} \neq 0, \\ 0 & \text{for } \alpha_{k+1} = 0, \end{cases}$$

$$\Gamma_k''(z) = (\Gamma_k(z) - \gamma_k)/(1 - \bar{\gamma}_k\Gamma_k(z)), \qquad \Gamma_k'(z) = \Gamma_k''(z)/\zeta_{k+1}(z),$$

$$\rho_k = \Gamma_k'(0) = \begin{cases} \gamma_k/\alpha_{k+1} & \text{for } \alpha_{k+1} \neq 0, \\ \lim_{z\to 0} \Gamma_k(z) & \text{for } \alpha_{k+1} = 0, \end{cases}$$

$$\Gamma_{k+1}(z) = \left(\Gamma_k'(z) - \rho_k\right)/\left(1 - \bar{\rho}_k\Gamma_k'(z)\right),$$
for $k = 0, 1, 2, \dots$

Similar to Theorem 2.2 we have that all $\Gamma_k(z)$ are Schur functions, normalized such that $\Gamma_k(0) = 0$ while all γ_k and ρ_k are in D. The algorithm stops if $|\rho_n| = 1$. For computational purposes, it is more useful to have an homogeneous reformulation in terms of acceptable functions. Suppose $\Delta^0(z)$ is the acceptable function of the previous section. Then compute

$$\gamma_k = \Delta_1^k(\alpha_{k+1})/\Delta_2^k(\alpha_{k+1}) \quad \text{if } \alpha_{k+1} \neq 0, \text{ otherwise } \gamma_k = 0,$$

$$\Delta^{k''}(z) = \Delta^k(z)(1 - |\gamma_k|^2)^{-1/2}\begin{bmatrix} 1 & -\bar{\gamma}_k \\ -\gamma_k & 1 \end{bmatrix}, \qquad \Delta^{k'}(z) = \Delta^{k''}(z)\begin{bmatrix} (\zeta_{k+1}(z))^{-1} & 0 \\ 0 & 1 \end{bmatrix},$$

$$\rho_k = \Delta_1^{k\prime}(0)/\Delta_2^{k\prime}(0) \quad (=\gamma_k/\alpha_{k+1} \quad \text{if } \alpha_{k+1} \neq 0),$$

$$\Delta^{k+1}(z) = \Delta^{k\prime}(z)\left(1-|\rho_k|^2\right)^{-1/2}\begin{bmatrix} 1 & -\bar{\rho}_k \\ -\rho_k & 1 \end{bmatrix},$$

for $k = 0, 1, 2, \ldots$

Again all the Δ^k are acceptable with $\Delta_1^k(0) = 0$.

One iteration can be summarized as

$$\Delta^{k+1} = \Delta^k \theta_k^{-1},$$

with

$$\theta_k = \left(1-|\rho_k|^2\right)^{-1/2}\begin{bmatrix} 1 & \bar{\rho}_k \\ \rho_k & 1 \end{bmatrix}\begin{bmatrix} \zeta_{k+1}(z) & 0 \\ 0 & 1 \end{bmatrix}\left(1-|\gamma_n|^2\right)^{-1/2}\begin{bmatrix} 1 & \bar{\gamma}_k \\ \gamma_k & 1 \end{bmatrix}.$$

Define as before

$$\Theta_0 = R_0^{-1/2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Theta_{n+1} = \theta_n \Theta_n \quad \text{for } n \geqslant 0.$$

To give its structure we have to introduce the concept of reproducing kernels and functions of the second kind.

Theorem 3.1. *Suppose* $\{\phi_k^*(z)\}$ *is an orthonormal basis for* $\mathcal{L}_n$ *in* $L_2(\omega)$, *then*

$$k_n(x, y) = \sum_{s=0}^{n} \overline{\phi_s^*(y)}\phi_s^*(x)$$

is a reproducing kernel for $\mathcal{L}_n$, *i.e.*

$$(p(z), k_n(z, y))_\omega = p(y) \quad \textit{for any } p(z) \in \mathcal{L}_n.$$

Proof. See [11]. □

If $f \in \mathcal{L}_n$, then the second kind function associated with f is the function $f^{\#} \in \mathcal{L}_n$ which interpolates $\tilde{\Omega}(z) f(z)$ in the points $0, \alpha_1, \alpha_2, \ldots, \alpha_n$ where $\tilde{\Omega}(z)$ is as defined before. We have:

Theorem 3.2. $f^{\#}$ *is the second kind function of* $f \in \mathcal{L}_n$ *iff* $f^{\#}$ *is the* $L_2(\mathrm{d}\theta)$ *projection of* $\tilde{\Omega}(z) f(z)$ *onto* $\mathcal{L}_n$.

Proof. See [18]. □

We have now the following structure for Θ_n:

Theorem 3.3.

$$\Theta_n = \tfrac{1}{2}\begin{bmatrix} \tilde{k}_n^*(z, 0) + \tilde{l}_n^*(z, 0) & \tilde{k}_n^*(z, 0) - \tilde{l}_n^*(z, 0) \\ \tilde{k}_n(z, 0) - \tilde{l}_n(z, 0) & \tilde{k}_n(z, 0) + \tilde{l}_n(z, 0) \end{bmatrix},$$

with $\tilde{k}_n(x, y) = k_n(x, y)\,(k_n(y, y))^{-1/2}$ *and* $\tilde{l}_n(x, y) = l_n(x, y)\,(l_n(y, y))^{-1/2}$. $k_n(x, y)$ *is a reproducing kernel for* $\mathcal{L}_n$ *in the space* $L_2(\omega)$. $l_n(x, y)$ *is the reproducing kernel for* $\mathcal{L}_n$ *in the space* $L_2(w)$, *where* $w(z) = \frac{1}{2}[W(z) + W_*(z)]$ *and* $W(z) = \Omega(z)^{-1}$. $\tilde{l}_n(z, 0)$ *is the second kind function for* $\tilde{k}_n(z, 0)$.

The upper star generalizes the concept of parahermitian conjugate. It is defined by

$$f^*(z) = U_n(z) f_*(z) \quad \text{for all } f \in \mathcal{L}_n.$$

Proof. See [6,7,8]. □

Note that $k_n(z, \alpha_n) = \phi_n^*(\alpha_n)\, \phi_n^*(z)$, thus for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$: $\tilde{k}_n(z, 0) = \kappa_n \phi_n^*(z)$ with $\phi_n(z)$ the nth Szegö polynomial.

The orthonormal functions of $\mathcal{L}_n$ as subspace of $L_2(w)$ are direct generalizations of the Szegö polynomials of the second kind. The analogs of the Szegö recursions become

$$\begin{bmatrix} \tilde{k}_{n+1}^*(z, 0) \\ \tilde{k}_{n+1}(z, 0) \end{bmatrix} = \theta_n \begin{bmatrix} \tilde{k}_n^*(z, 0) \\ \tilde{k}_n(z, 0) \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{l}_{n+1}^*(z, 0) \\ -\tilde{l}_{n+1}(z, 0) \end{bmatrix} = \theta_n \begin{bmatrix} \tilde{l}_n^*(z, 0) \\ -\tilde{l}_n(z, 0) \end{bmatrix},$$

with initial conditions $\tilde{k}_0(z, 0) = \tilde{l}_0(z, 0) = R_0^{-1/2}$. So that we have $k_n(0, 0) = R_0^{-1} \prod_{k=0}^{n-1}(1 - |\gamma_n|^2)/(1 - |\rho_k|^2)$.

It should be noted that the last step in the Nevanlinna–Pick algorithm which normalized $\Gamma_{n+1}(0) = 0$ is the so called Denjoy normalization. We could also have normalized so that $\Gamma_{k+1}(y) = 0$. This would then give the recursion for the general $\tilde{k}_n(x, y)$. See [4,8].

Theorem 2.4 has now an interpolation equivalent.

Theorem 3.4. (1) *Define* $\tilde{\Omega}_n(z) = \tilde{l}_n(z, 0)/\tilde{k}_n(z, 0)$ *and* $\tilde{\Omega}(z) = \Omega(z)/R_0$, *then* $\tilde{\Omega}_n(z)$ *interpolates* $\tilde{\Omega}(z)$ *in* $z = 0, \alpha_1, \alpha_2, \ldots, \alpha_n$ *in a multipoint Padé sense.*

(2) *Define* $\tilde{\omega}_n(z) = \frac{1}{2}[\tilde{\Omega}_n(z) + \tilde{\Omega}_{n*}(z)]$ *and* $\tilde{\omega}(z) = \omega(z)/R_0$, *then*

$$\tilde{\omega}_n(z) = R_0^{-1} U_n(z) / \big(\tilde{k}_n(z, 0) \tilde{k}_n^*(z, 0)\big)$$

is a $(0/n)$ *Laurent–Hermite interpolant for* $\tilde{\omega}(z)$ [6].

Proof. See [6,8]. □

Now we can give the solution to the problem given at the beginning of this section. We have:

Theorem 3.5. *The minimum* S_n *of*

$$\min_{f_n} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{\sigma(e^{i\theta})} - f_n(e^{i\theta}) \right|^2 \omega(e^{i\theta})\, d\theta,$$

where f_n *ranges over the rational functions in* $\mathcal{L}_n$ *is obtained for* $f_n = k_n(z, 0)\, \sigma(0)$ *and the minimum is given by*

$$S_n = 1 - k_n(0, 0) \sigma(0)^2.$$

Proof. See [8]. □

This concludes the parallel treatment of the Schur and Nevanlinna–Pick algorithms. In the next section we shall investigate how the transmission zeros $\alpha_1, \alpha_2, \ldots,$ have to be found.

4. The optimal location of the transmission zeros

It was shown in the previous section how ρ_{n-1} and γ_{n-1} depend on $\alpha_1, \alpha_2, \ldots, \alpha_n$. Thus also S_n depends on these parameters. Clearly S_n is minimized if we minimise

$$E_n = \prod_{s=0}^{n-1} \left(1 - |\rho_s(\alpha_1, \ldots, \alpha_{s+1})|^2\right) / \left(1 - |\gamma_s(\alpha_1, \ldots, \alpha_{s+1})|^2\right),$$

where α_k may range over D. They cannot be taken outside D because we then would obtain an unstable approximant. Thus we have to solve a constrained optimization problem. The fact that the α_k are constrained in D is however no problem because if $\alpha_{p+1} \in T$, then $(1 - |\rho_p|^2)/(1 - |\gamma_p|^2) = 1$, which means that there will be no reduction in S_p for such a choice of α_{p+1}, while any other value of α_{p+1} in D will give a better result. Thus the constraints will never be active for the optimal solution. The fact that $\alpha_k \in D$ can be used as a trust region for the optimization routine used, i.e. it can be used to bound the steplength in the iterative optimization process.

Because the computation of E_n is recursive in nature we could try to solve our optimization problem recursively. Unfortunately we do not have a consistency property. I.e.
If

$$E_n(\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0) \text{ is minimal}$$

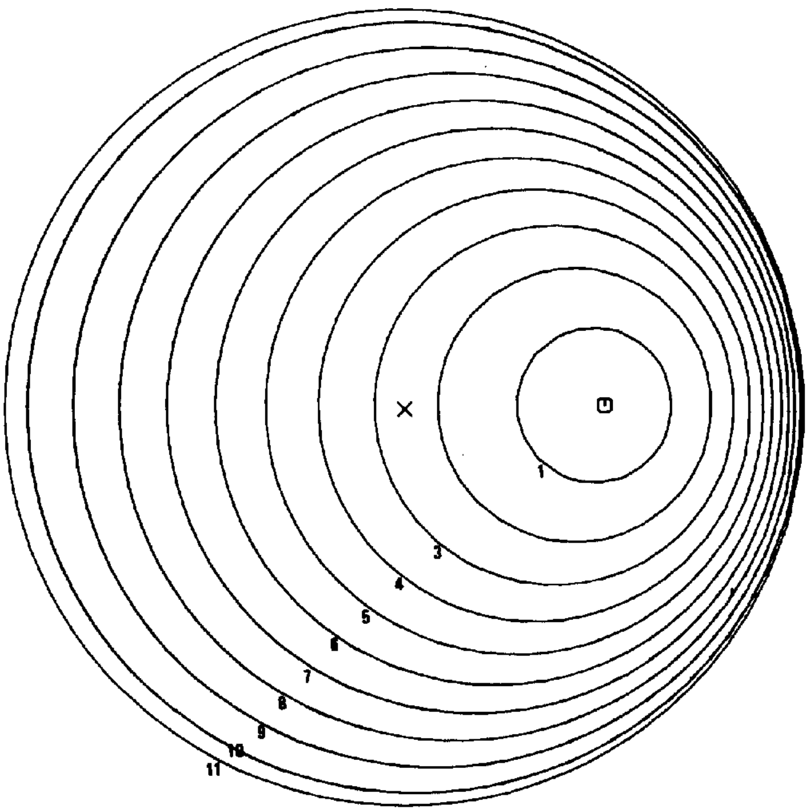


Fig. 1. $n = 1$, $\alpha_1 = 0.5$, $\beta_1 = 0.0$, $c_i = 0.81(0.02)0.99, 1.0$, min $= 0.8002$, $L = 0$, $M = 1$ (See also Appendix A).

and

$$\min_{\alpha_{n+1}} E_{n+1}(\alpha_1^0, \alpha_2^0, \ldots, \alpha_n^0, \alpha_{n+1}) \text{ is obtained for } \alpha_{n+1} = \alpha_{n+1}^0$$

then

$$E_{n+1}(\alpha_1^0, \alpha_2^0, \ldots, \alpha_{n+1}^0) \neq \min_{\alpha_1, \ldots, \alpha_{n+1}} E_{n+1}(\alpha_1, \ldots, \alpha_{n+1}).$$

Also it is not a good strategy to use $(\alpha_1^0, \ldots, \alpha_n^0, \alpha)$ with a $\alpha \in D$ as a starting point for the optimization problem in step $n+1$, because it could be quite different from the optimum for this problem.

What we do have is that if $\sigma(z)$ is rational of degree n, then the solution of min $E_m(\alpha_1, \ldots, \alpha_m)$ will theoretically give the exact solution for all $m \geqslant n$.

5. Examples

We shall now illustrate the nature of the previously described optimization problem with some very simple examples.

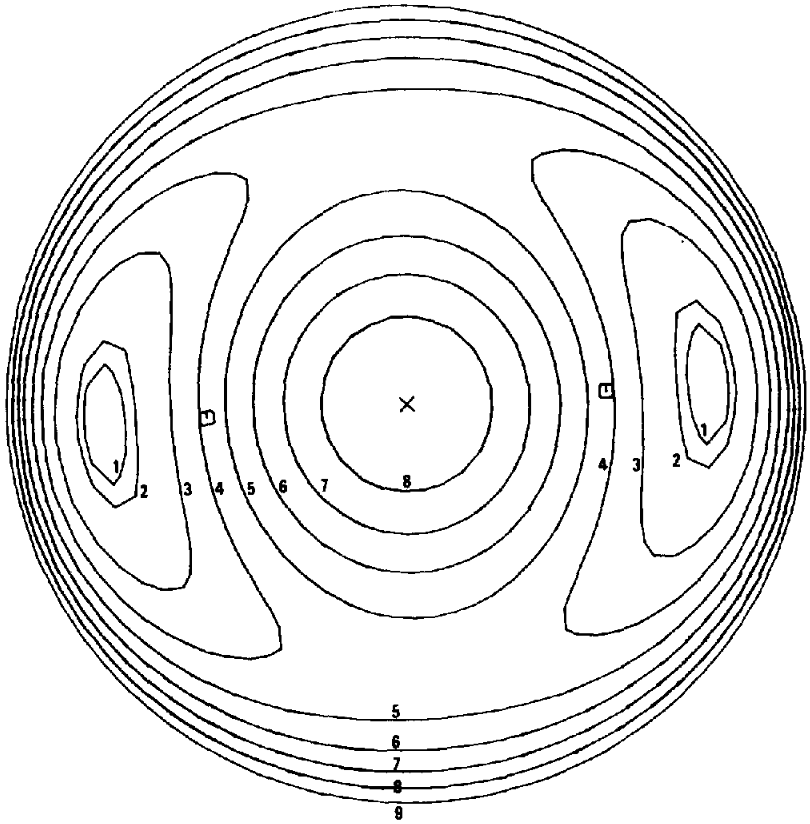


Fig. 2. $n = 2$, $\alpha_1 = -\alpha_2 = 0.5$, $\beta_1 = \beta_2 = 0.0$, $c_i = 0.982$, 0.9825(0.0025)1.0, min = 0.98151, $L = 0$, $M = 1$ (See also Appendix A).

First of all we restrict ourselves to real R_k and to functions $\sigma(z)$ that are rational. A consequence is that the TZ are real or appear in complex conjugate points.

Example 5.1. We suppose that $\sigma_*(z)$ has the form $(z-\alpha_1)/(z-\beta_1)$. Thus after one extraction, the minimum of $E_1(\alpha)=(1-|\rho_0(\alpha)|^2)/(1-|\gamma_0(x)|^2)$ must yield the correct value of α. In Fig. 1 we took $\alpha_1=0.5$ and $\beta_1=0$. Some contour lines for $E_1(\alpha)$, $\alpha \in D$ are given. $E_1(\alpha)=1$ for $\alpha \in T$ as will always be in the following examples. The minimum is obtained for $\alpha=0.5$ as it should be.

Example 5.2. We take $\sigma_*(z)=\prod_{i=1}^{2}(z-\alpha_i)/(z-\beta_i)$ with $\alpha_1=\alpha_2=0.5$ and $\beta_1=\beta_2=0$. After one step of the extraction we have plotted some contour lines of $E_1(\alpha)$ in Fig. 2. The figure is symmetric but the minima are different from ± 0.5.

Example 5.3. That the optimization of $E_1(\alpha)$ can give a quite misleading idea about the position of α_i is illustrated in this example. It is of the same form as Example 5.2 but with $\alpha_1=0.5$, $\alpha_2=0$ and $\beta_1=\bar{\beta}_2=0.5\exp(i\pi/4)$. Fig. 3 shows a minimum close to -0.5 which is the opposite of the exact α_1. Such fenomena have an explanation. The approximation is least squares on the unit circle. The effect of α_1 on the circle is neutralized by the two poles especially in the first and fourth quadrant. This causes the shift to the left of the TZ if there is only one zero allowed.

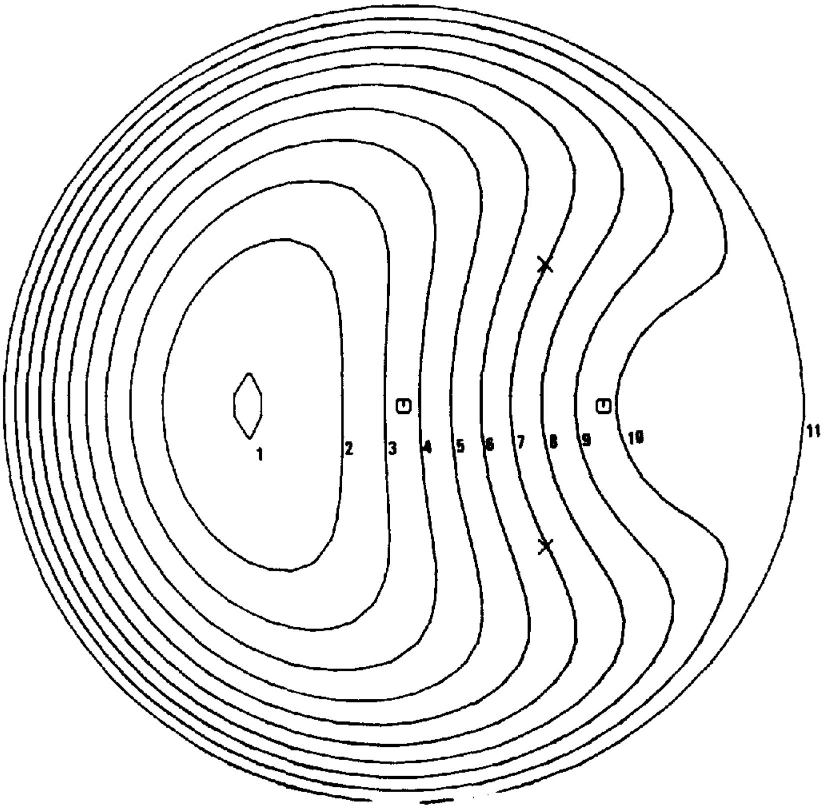


Fig. 3. $n=2$, $\alpha_1=0.5$, $\alpha_2=0.0$, $\beta_1=\bar{\beta}_2=0.5\exp(i\pi/4)$, $c_i=0.95(0.005)1.0$, min $=0.94982$, $L=0$, $M=1$ (See also Appendix A).

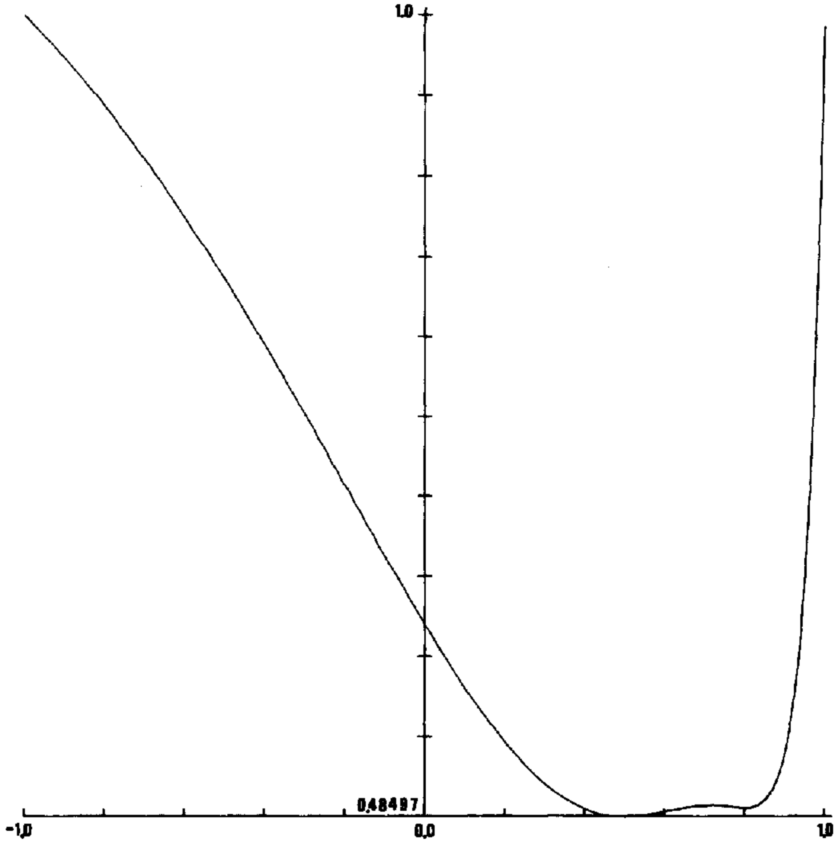
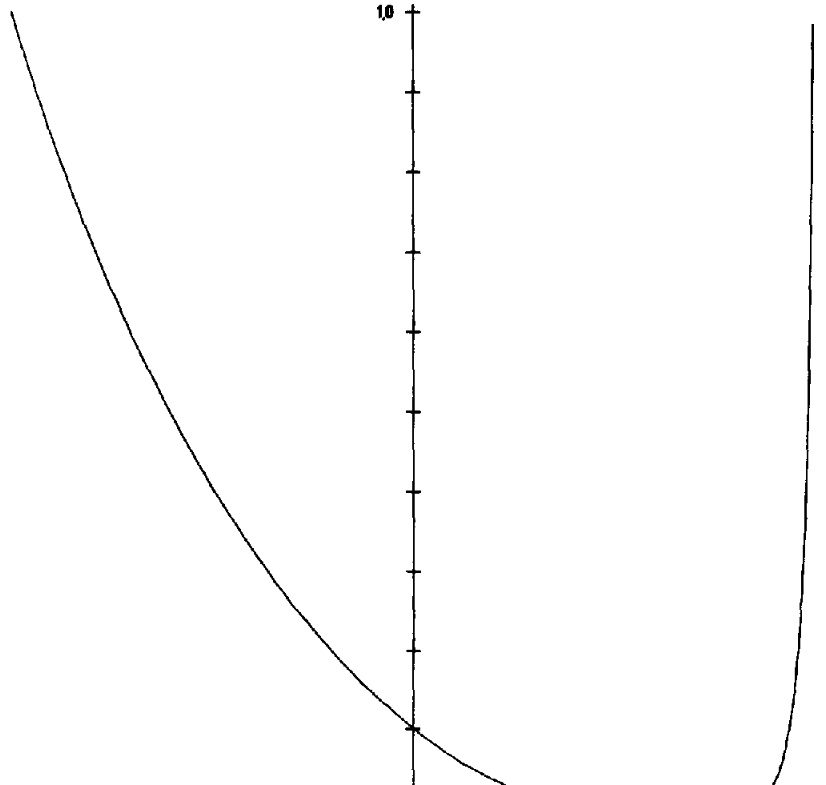


Fig. 4. $n = 2$, $\alpha_1 = \alpha_2 = 0.5$, $\beta_1 = \beta_2 = 0.0$, min = 0.48497, $L = 0$, $M = 2$ (See also Appendix A).



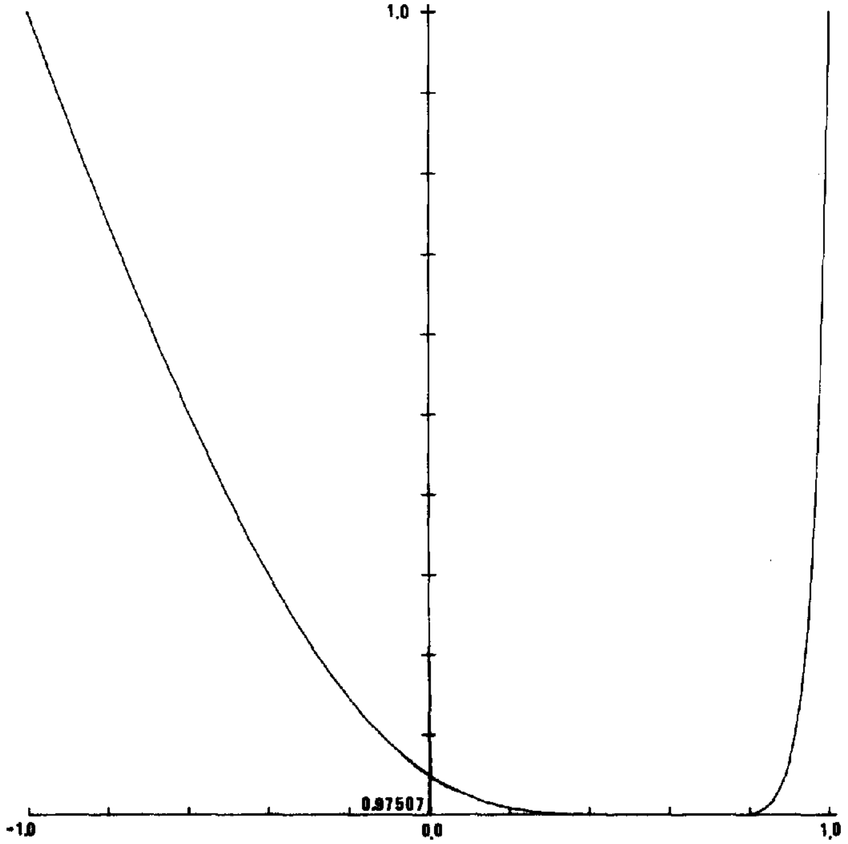
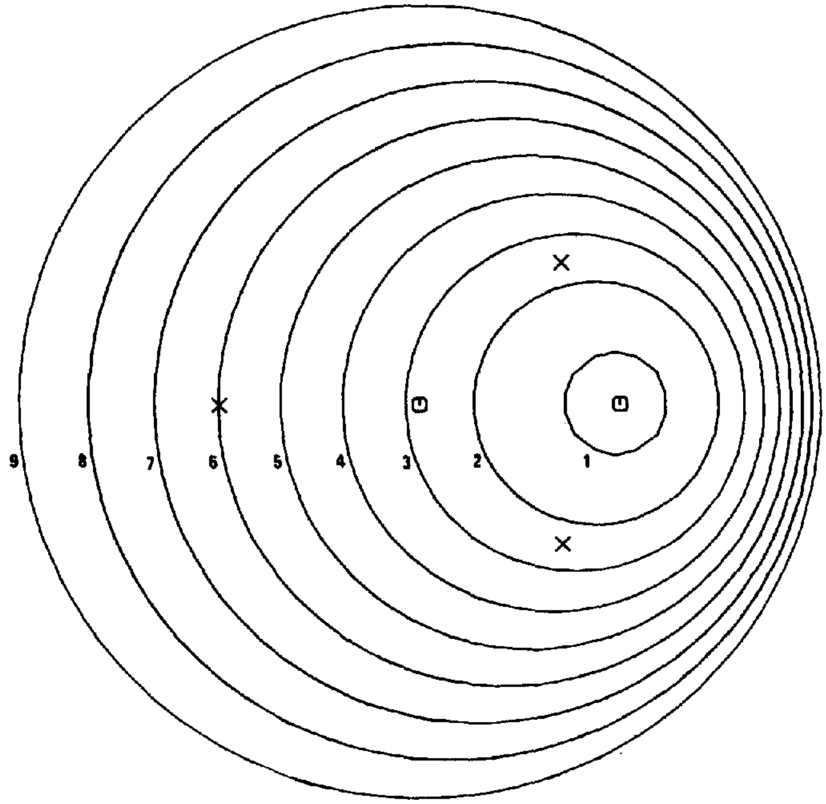


Fig. 6. $n = 4$, $\alpha_1 = \alpha_2 = 0.5$, $\alpha_3 = \alpha_4 = 0.0$, $\beta_1 = \bar{\beta}_2 = 0.5\ \exp(i\pi/4)$, $\beta_3 = \beta_4 = -0.5$, min = 0.97507, $L = 4$, $M = 3$ (See also Appendix A).



Example 5.4. Next we take a $\sigma_*(z)$ of degree n but suppose that only one TZ is nonzero. It may have a multiplicity $m \leqslant n$. Because R_k was real, this TZ must be real. We can find this zero as follows: first perform a number of Levinson extractions. At least $n-m$, but there may be more. After this we perform at least m extractions at a point $\alpha \in (-1,1)$. This results in a function E_k which only depends on α and which should be optimized in $(-1,1)$. The optimum must be obtained for the exact α. We illustrate this by taking $\sigma_*(z) = (z-0.5)^2/z^2$. No ordinary Levinson extractions are needed and Fig. 4 gives a plot of $E_2(\alpha)$ for $\alpha \in [-1,1]$. The minimum is as expected in $\alpha = 0.5$.

Example 5.5. The next two figures are related to the previous example, where $\sigma_*(z) = (\prod_1^4(z-\alpha_i)/(z-\beta_i)$ with $\alpha_1 = \alpha_2 = 0.5$, $\alpha_3 = \alpha_4 = 0$ and $\beta_1 = \beta_2 = -0.5$, $\beta_3 = \bar{\beta}_4 = 0.5\ \exp(i\pi/4)$. In Fig. 5 we plotted $E_4(\alpha)$ after two ordinary Levinson extractions and two extractions at α. In Fig. 6 four Levinson extractions and three extractions at α are done

Figs. 4–6 illustrate in a clear way the ill conditionning of the problem. Although the original $\sigma_*(z)$ are nicely behaving and of a rather simple structure, it is numerically almost impossible to find the optimum accurately.

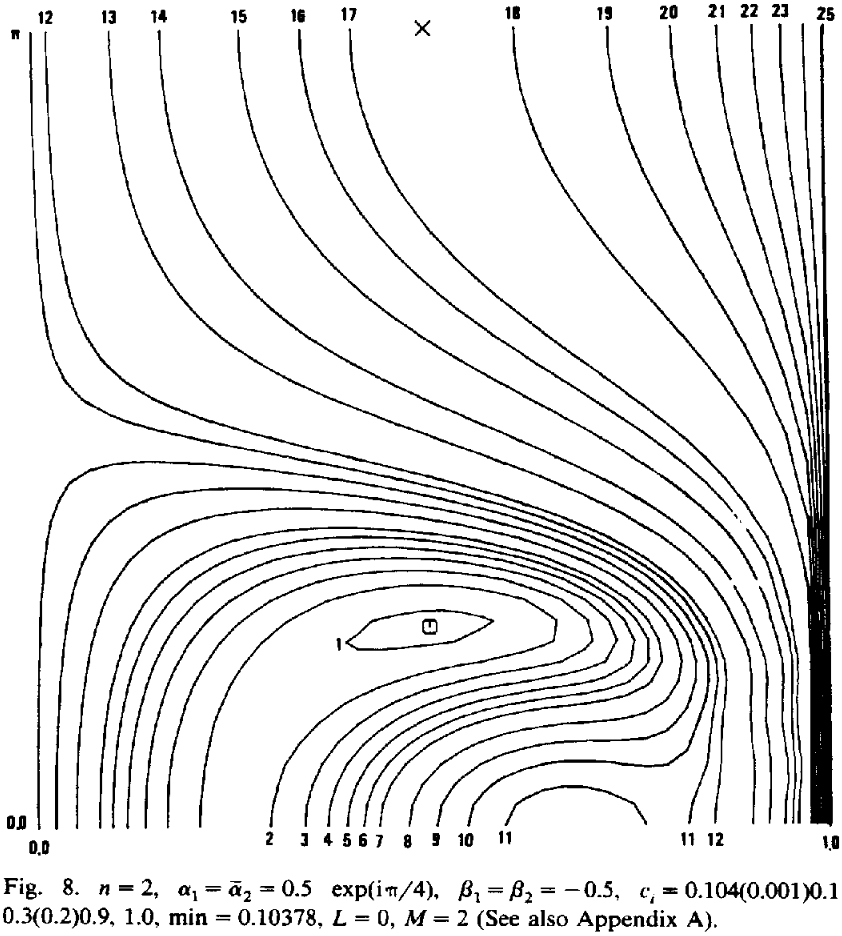


Fig. 8. $n=2$, $\alpha_1 = \bar{\alpha}_2 = 0.5\ \exp(i\pi/4)$, $\beta_1 = \beta_2 = -0.5$, $c_i = 0.104(0.001)0.11(0.002)0.12(0.01)0.14$, $0.14(0.02)0.2$, $0.3(0.2)0.9$, 1.0, min $= 0.10378$, $L = 0$, $M = 2$ (See also Appendix A).

Example 5.6. That this ill conditionning extends off the real line can be seen in Fig. 7. This represents some contour lines of $E_3(\alpha)$, $\alpha \in D$, obtained after two Levinson extractions and one extraction at $\alpha \in D$. for $\sigma_*(z) = \prod_1^3 (z - \alpha_i)/(z - \beta_i)$ with $\alpha_1 = 0.5$, $\alpha_2 = \alpha_3 = 0$ and $\beta_1 = 0.5$, $\beta_2 = \bar{\beta}_3 = 0.5 \exp(i\pi/4)$.

Example 5.7. Finally we consider the problem of extracting one conjugate pair of TZ. In this case a number of extractions at $\alpha = 0$, is followed by a pair of extractions at α and $\bar{\alpha}$. We put $\alpha = r \exp(i\theta)$ so that we have to optimize $E_k(\alpha) = E_k(r, \theta)$ where $r \in (0, 1)$ and $\theta \in (0, \pi)$. Fig. 8 gives such contour lines for $E_k(r, \theta)$ generated for $\sigma_*(z) = (z - \alpha)(z - \bar{\alpha})/(z - 0.5)^2$ with $\alpha = 0.5 \exp(i\pi/4)$, obtained after one pair of complex conjugate extractions. The minimum is at the expected place but again note the ill conditioning. Fig. 9 gives similar contour lines for $\sigma_*(z) = \prod_1^3 (z - \alpha_i)/(z - \beta_i)$ with $\alpha_1 = \bar{\alpha}_2 = 0.5 \exp(i\pi/4)$, $\alpha_3 = 0$ and $\beta_1 = \beta_2 = -0.5$, $\beta_3 = 0.8$. The extractions are at 0 and at α and $\bar{\alpha}$.

6. Conclusion

The Nevanlinna–Pick algorithm is resumed together with a review of related approximation results. The straightforward generalization of the classical Szegö–Levinson–Schur theory is

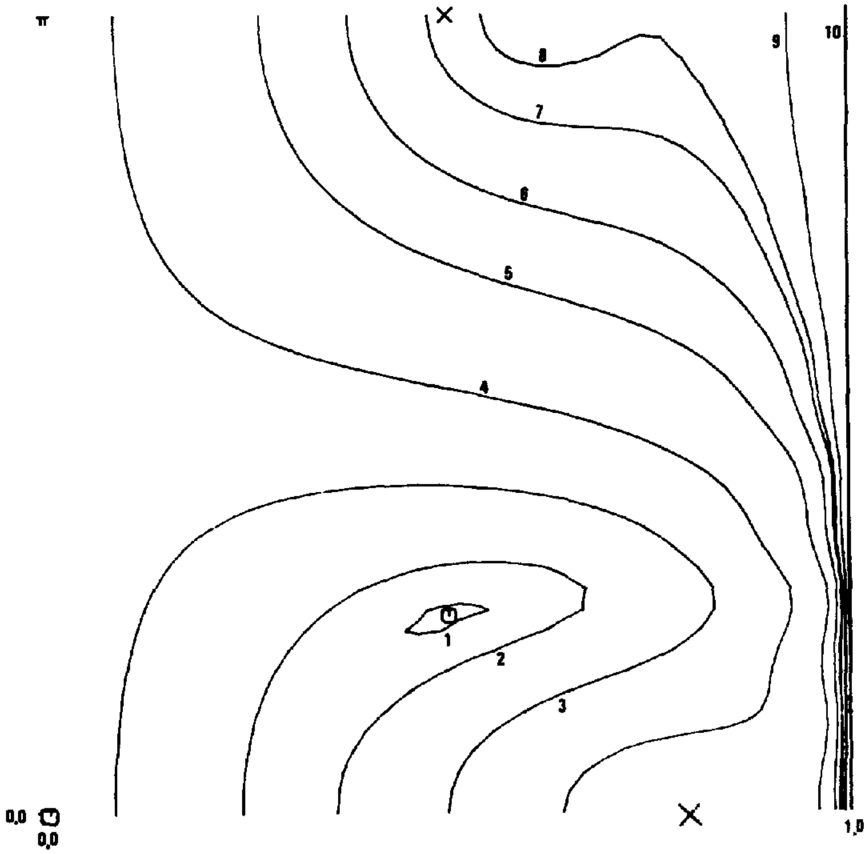


Fig. 9. $n = 3$, $\alpha_1 = \bar{\alpha}_2 = 0.5 \exp(i\pi/4)$, $\alpha_3 = 0.0$, $\beta_1 = \beta_2 = -0.5$, $\beta_3 = 0.8$, $c_i = 0.814$, 0.82, 0.85, 0.9, 0.95, 0.97, 0.98, 0.983, 0.99, 1.0, min $= 0.81361$, $L = 1$, $M = 2$ (See also Appendix A).

illustrated. The transmission zeros of a filter constructed by this algorithm are the interpolation points for the Nevanlinna–Pick algorithm. They can be chosen freely in the open unit disc. The least squares error will therefore depend on the choice of these points and can thus be further reduced if these points are optimally chosen. This optimization problem is however very ill conditioned, which was illustrated by some simple examples. There is a significant reduction in the least squares error compared with the AR model, but once the TZ are in the neighborhood of the exact ones, the reduction in the error is only marginal. Hence only a coarse estimate of the TZ is sufficient to obtain a good ARMA approximant.

It should be mentioned that other methods exist to compute the TZ. In [3] e.g. it is shown how the qd algorithm may be used to find them.

Appendix A

In this appendix, the figures are resumed. They all relate to a give filter of the form

$$\sigma_*(z)=\prod_1^n (z-\alpha_i)/(z-\beta_i).$$

n, α_i, and β_i are indicated for each figure. A plot is given of the objective function

$$S_M(\alpha)=\sum_i^M \left(1-|\rho_i(\alpha)|^2\right)/\left(1-|\gamma_i(\alpha)|^2\right),$$

obtained with M generalized extractions performed on the acceptable function resulting from L ordinary Levinson extractions at $\alpha=0$.

In Figs. 1–3 and 7, α ranges over the unit disc. In Figs. 4–6, $\alpha\in[-1,1]$, while in Figs. 8 and 9, $\alpha=r\exp(\mathrm{i}\theta)$ with $r\in[0,1]$ and $\theta\in[0,\pi]$.

For Figs. 1–3 and 7–9 contour lines are plotted for $S_M(\alpha)=c_i$. On the unit circle is $S_M(\alpha)=1$. The contour constants c_i are indicated for each figure.

Poles are shown by a cross and zeros by a circle.

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The problem of finding the transmission zeros was originally suggested to us by P. Dewilde. We had many valuable discussions on this subject.

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